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Field-theoretical renormalisation and fixed-point structure of a generalised Coulomb gas

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Abstract. We study the renormalisation aspects of a generalised Coulomb gas by means of a novel method of renormalisation. The model corresponds to an XY system with N-fold symmetry breaking perturbation and is cast in a field-theoretical framework.

The method uses the operator product expansion and the renormalisation constants are obtained from the singular terms in this expansion as the ultraviolet cutoff is removed. This method clearly exposes the multiplicative renormalisations involved. An expansion in the departure of the Kosterlitz-Thouless temperature, fugacities, and in $\varepsilon = N-4$ to third order is performed. We find a non-trivial self-dual fixed point at which the vortex and symmetry breaking fugacities are of order $\sqrt{\varepsilon}$.

The method also allows the identification of a new set of 'fixed-point' theories that do not undergo renormalisation.

1. Introduction

In this paper we study the critical properties of a generalised Coulomb gas by means of a novel field-theoretical method of renormalisation.

The theory studied corresponds to an XY model with N-fold symmetry breaking perturbation in the Coulomb gas representation proposed by José *et al* (1977) (hereafter referred to as JKKN), Kadanoff (1978), Nienhuis (1984, 1987) and Cardy and Parga (1980).

The model has been previously studied in its field-theoretical framework by Wiegmann (1978), Ohta and Jasnow (1979), Amit *et al* (1980), and in the context of random symmetry breaking perturbations by Houghton *et al* (1981). In its simplest form (without symmetry breaking perturbations) the associated field theory is the sine-Gordon model. The first studies of this theory were done by Mandelstam (1975) and Coleman (1975) in a different context.

To our knowledge the work of Amit *et al* presents the most thorough analysis of the sine-Gordon model to date. However, the results of these authors on symmetry breaking perturbations is incomplete and does not agree with those of JKKN.

In particular Amit *et al* claim that the symmetry breaking perturbations renormalise independently of the other operators of the theory and this is in contradiction with the renormalisation group equations of JKKN. In this paper we propose a novel method of renormalisation that allows a better understanding of the renormalisation aspects and clearly shows how symmetry breaking perturbations do modify the renormalisation group equations by affecting wavefunction and vortex fugacity renormalisation.

The method involved exploits the operator product expansion (OPE) and provides a simple and systematic way of renormalising the theory by looking at the singularities in the OPE of the interaction terms in the field-theoretical Hamiltonian.

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Some virtues of the method are that it explicitly exhibits the need for multiplicative renormalisation of the singular terms in the OPE, and explicitly maintains the discrete symmetries of the theory. This method is radically different from that advocated by Amit *et al* (1980) in the context of the sine-Gordon model (no symmetry breaking perturbations), but close in spirit (but also somewhat different) to that advocated by Lovelace (1986).

It has been recognised by José *et al* (1977) and Amit *et al* (1980) that near the Kosterlitz–Thouless (1973 hereafter referred to as (KT)) temperature ($\beta^2 = 8\pi$ in Amit *et al*) perturbations with $N < 4$ are relevant, $N = 4$ marginal and $N > 4$ irrelevant.

We carry out a double expansion in $\delta = \beta^2/8\pi - 1$ and $\varepsilon = N - 4$. Our method clearly shows that the symmetry breaking perturbations drastically change the renormalisation aspects of the theory and, in fact, they contribute to the renormalisation of wavefunction and vortex fugacity. Renormalisation group equations are obtained up to third order in the fugacities.

We find a non-trivial self-dual fixed point at which $\beta = \beta^* = 2\pi N$ and the fugacities are of order $\sqrt{\varepsilon}$, for $N > 4$.

In § 2 we review the Coulomb gas model corresponding to the XY model with N-fold symmetry breaking interactions, and we derive the underlying field theory following Wiegmann (1978). The material in this section is fairly well known but is included for self-consistency.

In § 3 we introduce the renormalisation scheme and by looking at the singularities in the OPE we prove multiplicative renormalisation.

Section 4 is devoted to the renormalisation procedure and the computation of the renormalisation constants and the RG beta functions. The duality properties are analysed in these beta functions. The existence of a non-trivial self-dual fixed point is pointed out and some properties of this fixed point are analysed.

In § 5 we use the method to show the existence of a set of 'fixed-point' theories.

Section 6 relates the Coulomb gas under study in the previous sections to fermionic models of generalised dimerised spin chains.

A conclusion summarises the results and poses further questions.

Three appendices are devoted to technical details.

2. The model

The model that we propose to study is basically the planar model with an N-fold symmetry breaking field in the form proposed by JKKN.

After a duality transformation and in the Villain approximation (see José *et al* (1977) for details) the partition function is

$$Z[K, Y_0, Y_N] = \prod_{\boldsymbol{R}} \int_{-\infty}^{\infty} \mathrm{d}\phi(\boldsymbol{R}) \sum_{n(R)} \sum_{m(R)} \exp\left[\sum_{\boldsymbol{R}} \left(-\frac{1}{2K} (\nabla\phi - N\boldsymbol{A})^2 + 2\pi \mathrm{i} m(R)\phi(R) + \ln y_0 m^2(R) + \ln y_N n^2(R)\right)\right]. \tag{2.1}$$

In the above expression the integration range of the variable ϕ has been extended to infinity. $\boldsymbol{A}$ is an integer-valued (gauge) vector field that lives on the links of the original lattice obeying the condition

$$\nabla x\boldsymbol{A}(R) = n(R) \tag{2.2}$$

where ∇x is the two-dimensional curl. In equation (2.1) the integer-valued fields $m(\boldsymbol{R})$ and $n(\boldsymbol{R})$ (living at the sites of the lattice), represent the vortices and symmetry breaking excitations respectively. The terms with y_0 and y_N correspond to the fugacities of both types of excitations and we have assumed a simple form for these terms following Kadanoff (1978) and Nienhuis (1984, 1986). The above form for the fugacities term assumes charge symmetry with respect to the sign of the 'charges' $n(\boldsymbol{R})$, $m(\boldsymbol{R})$.

The partition function (2.1) has the gauge symmetry

$$\begin{aligned} \phi(R) &\rightarrow \phi(R) + N\alpha(R) \\ A(R) &\rightarrow A(R) + \nabla\alpha(R) \end{aligned} \tag{2.3}$$

with $\alpha(\boldsymbol{R})$ being an integer-valued field. It is precisely this gauge symmetry that allows the extension of the integration range in $\phi(\boldsymbol{R})$, and that ensures that condition (2.2) completely defines the vector field A up to 'gauge transformations'.

As pointed out by Kadanoff (1978) the above theory can be looked at as a system of electric charges and magnetic monopoles in two dimensions, under a duality transformation $n(\boldsymbol{R}) \leftrightarrow m(\boldsymbol{R})$.

Integrating out the spin waves in (2.1), the partition function becomes (Nienhuis 1984, 1986)

$$\begin{aligned} Z[K, y_0, y_N] = Z_{\mathrm{SW}} \sum_{n,m} \exp\bigg[\sum_{r,r^1} \bigg(\frac{-N^2}{2K} n(r)G(r, r^1)n(r^1) - 2\pi^2 K m(r) G(r, r^1) m(r^1) \\ - \mathrm{i}Nn(r)\theta(r, r^1)m(r^1) + \ln y_0 m^2(r) + \ln y_N n^2(r) \bigg) \bigg] \end{aligned} \tag{2.4}$$

where $G(\boldsymbol{r}, \boldsymbol{r}^1)$ is the two-dimensional Green function and $\theta(\boldsymbol{r}, \boldsymbol{r}^1)$ is defined by the relations

$$\begin{aligned} G(\boldsymbol{r}, \boldsymbol{r}^1) &= -\frac{1}{4\pi} \ln \mu^2 [\boldsymbol{r} - \boldsymbol{r}^1|^2 \\ \Theta(\boldsymbol{r}, \boldsymbol{r}^1) &= -\tan^{-1}\left(\frac{y - y^1}{x - x^1}\right) \\ \square G(r, r^1) &= -\delta^2(\boldsymbol{r}) \end{aligned} \tag{2.5}$$

where y, x are the cartesian coordinates on the plane and μ is an infrared cutoff inversely proportional to the size of the system.

The functions $G(\boldsymbol{r})$ and $\Theta(\boldsymbol{r})$ are dual to each other and satisfy

$$\partial_\mu G(\boldsymbol{r}) = \frac{1}{2\pi} \varepsilon_{\mu\nu} \partial^\nu \Theta(\boldsymbol{r}) \tag{2.6}$$

with $\mu, \nu = 1, 2$ and $\varepsilon_{12} = -\varepsilon_{21} = +1$.

Now we prove that the partition function $Z[K, y_0, y_N]$ given by equation (2.4) can be exactly written as

$$\begin{aligned} Z[K, y_0, y_N] = \prod_{\boldsymbol{R}} \int_{-\infty}^{\infty} \mathrm{d}\phi(r) \sum_{n,m} \exp\bigg[-\int \mathrm{d}^2 r \bigg(\tfrac{1}{2}(\partial_\mu \phi)^2 - \mathrm{i}2\pi\sqrt{K}\, m(r)\phi(r) \\ + \mathrm{i}\frac{N}{\sqrt{K}} \varepsilon_{\mu\nu} \partial_\mu A_v(r) \tilde{\phi}(r) - \ln y_N n^2(r) - \ln y_0 m^2(r) \bigg) \bigg] \end{aligned} \tag{2.7}$$

with the definitions

$$\varepsilon_{\mu\nu} \partial_\mu A_v(r) = n(r) \tag{2.8a}$$

$$\varepsilon_{\mu\nu} \partial_v \tilde{\phi}(r) = \mathrm{i}\partial_\mu \phi. \tag{2.8b}$$

The field $\tilde{\phi}(r)$ is the dual to ϕ (Wiegmann 1978). The equivalence between (2.7) and (2.4) is proved as follows: integrating by parts the term with $\tilde{\phi}$ in (2.7), using (2.8b) and integrating by parts again it is written as

$$-\frac{N}{\sqrt{K}}\partial_\mu A_\mu \phi(r).$$

Now the Gaussian integrals over ϕ can be performed and by using equation (2.5) and the relation given by (2.6) one arrives at the expression given by (2.4).

Keeping only the excitations with m, $n=0$, ± 1 as in JKKN we approximate $Z[K, y_0, y_N]$ by

$$Z[\beta,\alpha,G]\approx\int \mathscr{D}\phi \exp\left\{-\int \mathrm{d}^2 r\left[\tfrac{1}{2}(\partial_\mu\phi)^2+\frac{\alpha_0}{\beta_0^2 a^2}\cos\beta_0\phi+\frac{G_0}{8\pi a^2}\cos\left(\frac{2\pi N}{\beta_0}\tilde{\phi}\right)\right]\right\} \tag{2.9}$$

where ϕ and $\tilde{\phi}$ are related as in equation (2.8b) and we introduced the lattice spacing (UV cutoff) a and defined

$$\begin{aligned}\beta_0&=\frac{2\pi}{\sqrt{K}}\\ 2y_0&=\alpha_0/\beta_0^2a^2\\ 2y_N&=G_0/8\pi a^2.\end{aligned} \tag{2.10}$$

The above definitions will allow us to compare the results with those of Amit *et al* (1980). The expression (2.9) is the starting point of the field-theoretical approach and coincides with that proposed by Wiegmann (1980).

3. OPE and multiplicative renormalisation

As usual we regard (2.9) as the continuation to Euclidean space of a quantum field theory in Minkowski space. The perturbative expansion and subsequent renormalisation will be carried out in the interaction picture in field theory in which the fields in the expansion are free. For the moment we will work in Minkowski space with signature $(+,-)$ and we will freely continue back to Euclidean space with $t=-\mathrm{i}y$. In two dimensions in Minkowski space the solutions to the wave equation

$$\Box\phi(x,t)=0 \tag{3.1}$$

are of the form

$$\phi(x,t)=\phi_\mathrm{R}(x-t)+\phi_\mathrm{L}(x+t) \tag{3.2a}$$

where ϕ_R, ϕ_L are right and left going waves. Out of ϕ_R and ϕ_L we construct the dual field

$$\tilde{\phi}(x,t)=\phi_\mathrm{R}(x-t)-\phi_\mathrm{L}(x+t) \tag{3.2b}$$

satisfying

$$\frac{\partial\phi}{\partial t}=-\frac{\partial\tilde{\phi}}{\partial x}. \tag{3.2c}$$

Therefore in Euclidean space with $t=-\mathrm{i}y$ they satisfy equation (2.8b).

The fields $\phi_R(x, t)$ and $\phi_L(x, t)$ are expanded in the usual way (see, for example, Green *et al* 1987) in terms of creation and annihilation operators and right and left going plane waves.

Defining the annihilation and creation parts of $\phi_R(\phi_L)$ as ϕ_R^+, ϕ_R^- (ϕ_L^+, ϕ_L^-), respectively, and defining

$$x - t = x^+ \qquad x + t = x^- \tag{3.3}$$

it is straightforward to find the following commutators:

$$\begin{aligned}
[\phi_R^+(x_1^+), \phi_R^-(x_2^+)] &= \langle \phi_R(x_1^+)\phi_L(x_2^+)\rangle = -\frac{1}{4\pi}\ln[\mu(x_1^+ - x_2^+)] \\
[\phi_L^+(x_1^-), \phi_L^-(x_2^-)] &= \langle \phi_L(x_1^-)\phi_L(x_2^-)\rangle = -\frac{1}{4\pi}\ln[\mu(x_1^- - x_2^-)].
\end{aligned} \tag{3.4}$$

In Euclidean space equations (3.3) become

$$x^+ = x + \mathrm{i}y = r\exp(\mathrm{i}\Theta) \qquad x^- = x - \mathrm{i}y = r\exp(-\mathrm{i}\Theta)$$

with r, Θ the polar coordinates on the plane. Therefore, the Euclidean continuation of equation (3.4) is written in terms of $G(r, r^1)$ and $\Theta(r, r^1)$ of equation (2.5). Again, we introduce the infrared cutoff $1/\mu$ of the order of the size of the system.

Using the results of appendix 1 we put into normal order the interaction terms in (2.9) with respect to the free massless field ϕ (in the interaction picture) and write the cosine terms in (2.9) as

$$\mathscr{L}_I = \frac{\alpha_0\mu^2}{\beta_0^2}(\mu^2 a^2)^{\delta_0}{:}\cos\beta_0\phi{:} + \frac{G_0\mu^2}{8\pi}(\mu^2 a^2)^{\tilde{\delta}_0}{:}\cos(2\pi N/\beta_0)\tilde{\phi}{:} \tag{3.5}$$

with

$$\begin{aligned}
\delta_0 &= \frac{\beta_0^2}{8\pi} - 1 \\
\tilde{\delta}_0 &= \frac{1}{8\pi}\left(\frac{2\pi N}{\beta_0}\right)^2 - 1
\end{aligned} \tag{3.6a}$$

and the double dots in (3.5) refer to the normal-ordering prescription of appendix 2. From expression (3.5) we see that the effective coupling constants are

$$\begin{aligned}
\alpha_1 &= \alpha_0(\mu^2 a^2)^{\delta_0} \\
G_1 &= G_0(\mu^2 a^2)^{\tilde{\delta}_0}.
\end{aligned} \tag{3.6b}$$

We now prove that all operators in the action (2.9) renormalise multiplicatively. For this we look at the operator product expansion of the normal ordered interaction terms and examine the coefficients of the operators that appear, identifying those that are singular in the limit $a \to 0$. The perturbative expansion is in terms of α_1, G_1, δ and $\tilde{\delta}$ since for δ, $\tilde{\delta} = 0$ the cosine operators in (3.5) become marginal (Amit *et al* 1980). This in turn means that, besides expanding in terms of the fugacities, we are also expanding in β_0^2 near $\beta_0^2 = 8\pi$, i.e. the KT temperature $K = 2/\pi$ and $\varepsilon = N - 4$.

Using the results of the appendices we find the following operator product expansions contributing to the renormalisation of wavefunction and interactions. In the following we cast our results in Euclidean space.

3.1. *Wavefunction renormalisation*

From appendix 2 we find

$:\cos\beta_0\phi(x)::\cos\beta_0\phi(y):$

$$=\frac{1}{4}\frac{1}{[\mu^2|x-y|^2]^{\beta_0^2/4\pi}}\left[2-\frac{1}{2}(\partial_\mu\phi(x))^2\beta_0^2|x-y|^2+\ldots\right] \tag{3.7}$$

where the dots stand for higher-order derivative terms that do not have singularities as $x\to y$ for $\beta_0^2\approx 8\pi$. Therefore, to second order in α_1, the term contributing to wavefunction renormalisation is

$$A=\left(-\frac{1}{2}(\partial_\mu\phi(x))^2\right)\frac{1}{8}\frac{\alpha_1^2\mu^2}{\beta_0^2}\int d^2y\frac{1}{[\mu^2|x-y|^2]^{1+2\delta_0}}. \tag{3.8}$$

Similarly the contribution to wavefunction renormalisation from the symmetry breaking perturbations is (see appendix 2)

$$B=\left(-\frac{1}{2}(\partial_\mu\phi(x))^2\right)\left[-\frac{1}{8}\frac{G_1^2\mu^2}{64\pi^2}\left(\frac{2\pi N}{\beta_0}\right)^2\right]\int d^2y\left(\frac{1}{\mu^2|x-y|^2}\right)^{1+2\tilde{\delta}_0}. \tag{3.9}$$

3.2. *Renormalisation of the vortex fugacity* (α/β_0^2)

To order α^3 the contribution arises from the operator product expansion of

$$\int d^2x\, d^2y\, d^2z :\cos\beta\phi(x)::\cos\beta\phi(y)::\cos\beta\phi(z):$$

$$=\frac{3}{4}\int d^2x\, d^2y\, d^2z :\cos\beta(\phi(x)+\phi(y)-\phi(z)):\left(\frac{\mu^2|x-y|^2}{\mu^2|x-z|^2\mu^2|y-z|^2}\right)^{\beta^2/4\pi}$$
$$+\text{non-singular terms.} \tag{3.10}$$

We must only keep the connected part of this contribution, since for $\beta^2\simeq 8\pi$ there is a quadratic divergence for both $x\to z$ and $y\to z$, but this is disconnected. The connected divergences have only logarithmic singularities (see appendix 3 for details). Therefore the contribution to the renormalisation of α_1/β_0^2 is given by the connected singularities of the expression above:

$$C=-\frac{1}{8}\frac{\alpha_1^3\mu^2}{(\beta_0^2)^3}\int d^2x :\cos\beta\phi(x):\int\mu^4\, d^2y\, d^2z\left(\frac{\mu^2|x-y|^2}{\mu^2|x-z|^2\mu^2|y-z|^2}\right)^{\beta^2/4\pi}. \tag{3.11}$$

The symmetry breaking perturbations *also* contribute to the renormalisation of α/β^2 as can be seen from the term

$$\frac{1}{2}\frac{G_1^2\mu^4}{(8\pi)^2}\left(-\frac{\alpha_1\mu^2}{\beta_0^2}\right)\int d^2x\int d^2y\int d^2z :\cos\gamma\tilde{\phi}(x)::\cos\gamma\tilde{\phi}(y)::\cos\beta\phi(z):$$

where $\gamma=2\pi N/\beta_0$. To this order we can set $N=4$ and $\beta_0^2\simeq 8\pi$. The singular part in this OPE contributes

$$D=\frac{1}{4}\frac{G_1^2}{(8\pi)^2}\frac{\alpha_1\mu^2}{\beta_0^2}\int d^2z :\cos\beta\phi(z):\int\mu^4\, d^2x\, d^2y$$
$$\times\left(\frac{1}{(\mu^2|x-y|^2)}\right)^{\beta^2/4\pi}\left(\frac{(z^+-x^+)(z^--y^-)}{(z^--x^-)(z^+-y^+)}\right)^{\beta^2/4\pi}. \tag{3.12}$$

Here again, only the connected singularities must be kept.

The above expression is a result of the 'angle' interaction of the 'electric' and 'magnetic' charges in (2.4).

3.3. Renormalisation of G_l

In the renormalisation of G_1 there is a term of order G_1^3 whose contribution is similar to the integral in C (equation (3.11)) and also a contribution of the form $(\alpha_1^2/\beta_0^4)\ G_1$ with the same singular structure as in D above (equation (3.12)). The details of these OPE are contained in appendix 2.

In order to regulate the singularities in the integrals we adopt the regularisation prescription that distances $|x-y|$, etc, are bounded in the interval

$$a<|x-y|<1/\mu$$

with a the underlying lattice spacing and $1/\mu$ the size of the system.

This prescription effectively introduces an IR and UV cutoff maintaining the dilatation symmetries of the theory in the sense that large and small distances are treated equivalently. The singularities are of the form $\ln \mu^2 a^2$, $\ln^2 \mu^2 a^2$, etc, as they must be in a scale-invariant theory.

This regularisation prescription is different from that of Amit *et al* (1980). These authors introduce a mass term as a 'soft' symmetry breaking perturbation.

The above tedious and technical manipulations were required to show via the OPE that the operators involved are renormalised multiplicatively. As usual, however, the free energy needs a subtraction corresponding to the disconnected singularities.

We can now proceed with the renormalisation prescription.

4. Renormalisation

After proving that the operators in the action renormalise multiplicatively we define the renormalised quantities and renormalisation functions as

$$\phi_0^2=\phi_R^2 Z_\phi \tag{4.1a}$$

$$\beta_0^2=\beta_R^2 Z_\phi^{-1} \tag{4.1b}$$

$$\cos\beta_R\phi_R=(\mu^2a^2)^{1+\delta_R}{:}\cos\beta_R\phi_R{:} \tag{4.1c}$$

$$\alpha_0 Z_\phi(\mu^2a^2)^{\delta_R}Z_1=\alpha_R \tag{4.1d}$$

$$G_0\tilde{Z}_1(\mu^2a^2)^{\tilde{\delta}_R}=G_R \tag{4.1e}$$

$$\delta_R=\beta_R^2/8\pi-1 \tag{4.1f}$$

$$\tilde{\delta}_R=\frac{(2\pi N)^2}{8\pi\beta_R^2}-1 \tag{4.1g}$$

$$\tilde{\phi}_0^2=\tilde{\phi}_R^2 Z_\phi^{-1} \tag{4.1h}$$

$$\cos\left(\frac{2\pi N}{\beta_R}\tilde{\phi}_R\right)=(\mu^2a^2)^{1+\tilde{\delta}_R}{:}\cos\left(\frac{2\pi N}{\beta_R}\tilde{\phi}_R\right){:}. \tag{4.1i}$$

In the above formula the dots refer to the normal ordering in the interaction picture of the (massless) renormalised field ϕ_R.

Notice the peculiar renormalisation of $\tilde{\phi}$ given by (4.1h) compared to (4.1a). This is, in fact, a consequence of the non-local relation (2.8b). In Minkowski space (see the discussion in § 3 and equation (3.3c)) $\tilde{\phi}$ is related to the *canonical momentum conjugate to* ϕ. Therefore, in order to maintain the canonical commutation relations, the renormalisations (4.1*a*, *h*) represent a Bogoliubov transformation to the renormalised fields and canonical momenta. The reader can be convinced of this fact by going to the Hamiltonian formulation in Minkowski space.

We now write the Lagrangian in (1.9) in terms of the renormalised quantities as

$$\mathscr{L}=\frac{1}{2}:(\partial_\mu\phi_R)^2:+\frac{\alpha_R\mu^2}{\beta_R^2}:\cos\beta_R\phi_R:+\frac{G_R\mu^2}{8\pi}:\cos\left(\frac{2\pi N}{\beta_R}\tilde{\phi}_R\right):$$
$$+\tfrac{1}{2}:(\partial_\mu\phi_R)^2:(Z_\phi-1)+\frac{\alpha_R\mu^2}{\beta_R^2}(Z_1^{-1}-1):\cos\beta_R\phi_R:$$
$$+\frac{G_R\mu^2}{8\pi}(\tilde{Z}_1^{-1}-1):\cos\left(\frac{2\pi N}{\beta_R}\right)\tilde{\phi}_R:. \tag{4.2}$$

The point now is that the counterterms are required to cancel the singularities in the OPE, and they are as usual considered as part of the interaction part of $\mathscr{L}$.

The new feature brought about by the counterterm Lagrangian is that the term $:(\partial_\mu\phi_R)^2:$ $(Z_\phi-1)$ contributes to the renormalisation of the fugacities.

To see this, consider the following OPE (see the appendices for details):

$$-\tfrac{1}{2}(Z_\phi-1)\int d^2x:(\partial_\mu\phi_R(x))^2:\left(-\frac{\alpha_R\mu^2}{\beta_R^2}\right)\int d^2y:\cos\beta_R\phi_R(y):$$
$$=\left(-\frac{\alpha_R\mu^2}{\beta_R^2}\right)(Z_\phi-1)\frac{2\beta^2}{(4\pi)^2}\int d^2y:\cos\beta_R\phi_R(y):\int\frac{d^2y}{|x-y|^2}+\text{non-singular.} \tag{4.3}$$

Since $Z_\phi-1\simeq O(\alpha^2,G^2)$ to this order we can set $\beta_R^2=8\pi$.

A similar contribution arises from the $:(\partial_\mu\phi_R)^2:(Z_\phi-1)$ term to the renormalisation of G.

With the results given by equations (3.8)-(3.11), (4.3) and a similar term for G we find after some straightforward algebra the following renormalisation functions in terms of the bare quantities α_0, G_0, δ_0, $\tilde{\delta}_0$ (up to third order):

$$Z_\phi=\left(1+\frac{\alpha_0^2I}{64}-\frac{\alpha_0^2\delta_0I}{64}+\frac{\alpha_0^2\delta_0}{64}I^2-\frac{G_0^2I}{64}-\frac{G_0^2\tilde{\delta}_0}{64}I-\frac{G_0^2\tilde{\delta}_0}{64}I^2\right) \tag{4.4a}$$

$$Z_1=\left(1-\frac{\alpha_0^2}{128}I^2+\frac{G_0^2I^2}{128}-\frac{G_0^2}{128}I\right) \tag{4.4b}$$

$$\tilde{Z}_1=\left(1-\frac{G_0^2}{128}I^2+\frac{\alpha_0^2}{128}I^2-\frac{\alpha_0^2}{128}I\right) \tag{4.4c}$$

with $I=\ln\mu^2a^2$.

With the definitions (4.1*a*–*h*) and the renormalisation constants (4.4*a*–*c*) we find the following RG beta functions:

$$\beta_\alpha=\frac{\partial\alpha_R}{\partial\ln\mu}=2\delta_R\alpha_R+\alpha_R\left(\frac{\alpha_R^2}{32}-\frac{G_R^2}{32}\right)-\frac{\alpha_RG_R^2}{64}+\ldots \tag{4.5a}$$

$$\beta_\beta=\frac{\partial\beta_R^2}{\partial\ln\mu}=2\beta_R^2\left(\frac{\alpha_R^2}{64}-\frac{G_R^2}{64}-\frac{\alpha_R^2\delta_R}{64}-\frac{G_R^2\tilde{\delta}_R}{64}+\ldots\right) \tag{4.5b}$$

$$\beta_G=\frac{\partial G_R}{\partial\ln\mu}=2G_R\tilde{\delta}_R-\frac{G_R\alpha_R^2}{64}+\ldots \tag{4.5c}$$

$$\beta_\delta=\frac{\partial\delta_R}{\partial\ln\mu}=\frac{\alpha_R^2}{32}-\frac{G_R^2}{32}\left(\frac{N^2}{16}\right)+\ldots \tag{4.5d}$$

where the dots stand for higher-order terms. Two special cases of the above equations can be considered with the purpose of comparison with previous results.

4.1. $G_R = 0$ (*no symmetry breaking perturbations*)

The above equations coincide with those of Lovelace (1986). The discrepancy between these equations and those of Amit *et al* (1980) has been discussed by Lovelace. To lowest order for $G_R = 0$ these are the celebrated Kosterlitz-Thouless (1973) equations.

4.2. $G_R \neq 0$

The equations in this case acquire a more symmetrical form by defining the vortex fugacity as

$$Y_R/8\pi = \alpha_R/\beta_R^2 \tag{4.6}$$

and similarly for the bare quantities. In terms of this new variable (4.5*a*–*d*) became

$$\beta_y = 2Y_R\delta_R - Y_R G_R^2/64 \tag{4.7a}$$

$$\beta_G = 2G_R\tilde{\delta}_R - G_R Y_R^2/64 \tag{4.7b}$$

$$\beta_\beta = \tfrac{1}{32}\beta_R^2[Y_R^2(1+\delta_R) - G_R^2(1+\tilde{\delta}_R)]. \tag{4.7c}$$

Notice that the above equations reflect the duality property $Y \leftrightarrow G$, $\delta \leftrightarrow \tilde{\delta}$. The relative minus sign in (4.7*c*) is a consequence of the relation (2.8*b*). The above equations for β_y and β_G seem in disagreement with those obtained by Nienhuis (1987) by using a Kosterlitz-Thouless renormalisation procedure.

Neglecting the terms YG^2 and GY^2 in (4.7*a*, *b*) reproduce the equations obtained by José *et al* (1977). Up to this order (without the YG^2 and GY^2 terms) there is the fixed line predicted by José *et al* beginning at the KT point

$$\beta_R^2 = 8\pi\ (\delta_R = 0) \qquad N = 4\ (\tilde{\delta}_R = 0)$$

$$Y_R = \pm G_R.$$

However the full beta functions (4.7*a*–*c*) have a non-trivial self-dual fixed point at

$$\beta_R^{2*} = 2\pi N \tag{4.8a}$$

$$\delta_R^* = \tilde{\delta}_R^* = \tfrac{1}{4}\varepsilon \tag{4.8b}$$

$$Y_R^{2*} = G_R^{2*} = 32\varepsilon \tag{4.8c}$$

with $\varepsilon = N - 4 > 0$.

This is one of the results of this paper. Notice that the perturbative expansion is in terms of $Y/8\pi$ and $G/8\pi$. Therefore, the fixed point (4.8*c*) is qualitatively within the perturbative regime even for $\varepsilon \approx O(1)$.

An analysis of the eigenvalues of the system of equations (4.7*a*–*c*) near the non-trivial fixed point (4.8*a*–*c*) shows that there is at least one irrelevant eigenvector, but in general RG trajectories are driven away from this fixed point in the infrared.

Theories on the lines $Y_R = \pm G_R$, $\beta_R = \beta_R^*$ for $N > 4$ are driven towards the KT fixed point ($\beta_{KT} = 8\pi$, $Y = G = 0$) for Y_R^2, $G_R^2 < 32\varepsilon$ or away from it for Y_R^2, $G_R^2 > 32\varepsilon$.

For $N < 4$ there is no non-trivial fixed point and at weak coupling the physical picture coincides with that of JKKN. At $N = 4$ there are now the JKKN fixed lines described by JKKN at $\beta^2 = 8\pi$ with $Y = \pm G$ and again the JKKN description is still valid. However, for $N > 4$ there are the new fixed points (four of them) that describe a new phase of the theory. These fixed points seem to describe a Z_N type or clock models (parafermionic theories) but in order to confirm this conjecture we need to calculate the conformal anomaly at this point. This is currently underway.

5. More fixed-point theories

The techniques developed in the previous two sections allow us to investigate theories more general than those described by the generalised Coulomb gas given by equations (1.4) and (1.9). In particular, consider actions of the form

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + Y\mu^2{:}\cos \beta\phi_R{:} + G\mu^2{:}\cos\beta\phi_L{:} \tag{5.1}$$

with ϕ_R and ϕ_L defined by equations (3.2*a*, *b*) and the double dots again indicating normal ordering in the interaction picture of the free massless field ϕ.

From the results of the appendices we find that the relevant contributions from the OPE to wavefunction and coupling constants renormalisation vanish upon angular integrations for $\beta^2 = 4\pi n$ with n = integer.

Hence, for $\beta^2 = 8\pi$ these are fixed-point theories for any values of Y and G.

In particular an 'action' of the form (5.1) has been obtained as a bosonic form of the critical Ising model by bosonisation of a free massless Majorana fermion by Boyanovsky (1988) and previously proposed by Kiritsis (1987) and Ogilvie (1981).

Notice, however, that critical models described by (5.1) cannot be obtained from the generalised Coulomb gas (2.4) and (2.9).

As in the case of the critical Ising model, we expect these fixed-point theories to describe some conformal invariant field theory providing, perhaps, an explicit realisation of representations of the Virasoro algebra. Of course, the question that remains is to understand the universality classes described by these theories. These questions will be addressed elsewhere.

6. Relation to fermion models

It is well known that the generalised Coulomb gas can be mapped onto fermion gas models and quantum spin chains (see, for example, Emery 1979).

The work of Black and Emery (1981) summarises the critical properties of some two-dimensional models and makes use of the bosonisation mapping of these models to a generalised Fermi gas (dimerised spin chain).

Now we use this mapping to find which fermion models can be studied from the Coulomb gas (2.9) by means of the RG equations obtained in the previous sections.

From the usual bosonisation rules (see Emery 1979, Banks *et al* 1976, Mandelstam 1975, Coleman 1975) it is found that

$$\mathrm{i}{:}(\psi_R^+\partial_x\psi_R - \psi_L^+\partial_x\psi_L){:} = (\partial_x\phi_R)^2 + (\partial_x\phi_L)^2$$

$$:\psi_R^+\psi_R: = \frac{1}{\sqrt{\pi}}{:}\partial_x\phi_R{:} \qquad :\psi_L^+\psi_L: = \frac{1}{\sqrt{\pi}}\partial_x\phi_L$$

$$:\psi_R^+\psi_L: = :\exp(\mathrm{i}\sqrt{4\pi}\,\phi):$$

$$[:\psi_R^+\psi_L^+:]^M = :\exp(\mathrm{i}M\sqrt{4\pi}\,\tilde{\phi}):$$

with ψ_R, ψ_L the right and left moving components of a free Fermi field.

Hence the Hamiltonian $H = H_1 + H_2$ with

$$H_1 = v_F \int \mathrm{d}x\{:i(\psi_R^+\partial_x\psi_R - \psi_L^+\partial_x\psi_L): + g_1((:\psi_R^+\psi_R:)^2 + (:\psi_L^+\psi_L:)^2) + \tfrac{1}{2}g^2{:}\psi_R^+\psi_R{:}{:}\psi_L^+\psi_L{:}\}$$

$$H_2 = \int \mathrm{d}x\{g_3(:\psi_R^+\psi_L + \mathrm{HC}:) + g_4[(:\psi_R^+\psi_L^+:)^M + (:\psi_L\psi_R:)^M]\}$$

is equivalent to the Coulomb gas (2.9), after a Bogoliubov transformation and a rescaling of the Fermi velocity

$$v_F \to \tilde{v}_F = v_F[(1+g_1/\pi)^2-(g_2/\pi)^2]^{1/2}.$$

In this language the constant β in (2.9) is given in terms of g_1, g_2 by

$$\beta = \sqrt{4\pi}\left(\frac{1+g_1/\pi-g_2/\pi}{1+g_1/\pi+g_2/\pi}\right)^{1/4}$$

and $M = N/2$.

We therefore see that the Coulomb gas described by (2.9) only applies to the Black-Emery Hamiltonian for $N = 2$, $D^+ = D^-$ (in BE convention) corresponding to the charge symmetric situation, and no umklapp scattering term ($g_\mu = 0$ in BE). The umklapp scattering term corresponds to keeping electric charges ±2 in the Coulomb gas description.

The application of the RG methods developed in this paper to the case of charge asymmetry and umklapp scattering, along with the study of susceptibilities and correlation functions, will be reported elsewhere.

7. Conclusions and further questions

In this paper we have studied a novel method of renormalisation in a field-theoretical description of a generalised Coulomb gas model. The method exploits the operator product expansion and allows a systematic and simple analysis of the renormalisation procedure. In particular, it explicitly shows that the operators in the original action are multiplicatively renormalised.

We consistently carried out a double expansion around the Kosterlitz-Thouless temperature and $\varepsilon = N-4$ with N representing an N-fold symmetry breaking perturbation to the XY model in the Coulomb gas formulation.

The expansion was carried out to third order and we found a non-trivial fixed point at which the fugacities are of order $\sqrt{\varepsilon}$ for $N>4$.

This method allowed us to identify a new set of 'fixed point' theories that do not undergo infinite renormalisations and do not seem *a priori* to be 'trivial'.

The method can be easily extended to incorporate charge asymmetry—this issue is currently under investigation.

One of the outstanding questions is the following: at the new fixed points the theory is conformally invariant and is therefore characterised by the value of the conformal anomaly c (Belavin *et al* 1984*a*, *b*, Friedan *et al* 1984, Cardy 1987). The first question is the value of c at the non-trivial fixed point.

Zamolodchikov's theorem (Zamolodchikov 1986) tells us that there exists a function c that interpolates between the conformal anomaly at the fixed points and that diminishes or is stationary along infrared flows. This then guarantees that, if an IR fixed point is reached along the IR flow, the value of c at the new fixed point is less or equal than that at the original theory.

The analysis presented at the end of § 5 indicates that on the planes $\delta = \tilde{\delta} = \delta^*$, the RG trajectories along the fixed lines $Y = \pm G$ flow towards the trivial fixed point with $c = 1$ in the IR. If this is indeed the case then this would suggest that the new fixed points describe theories with $c > 1$. We rule out the possibility for $c = 1$ since this would indicate that the perturbations just change the critical exponents but not the critical behaviour, but this is only consistent with marginal operators at the KT point. Therefore we conjecture that these fixed points theories have $c > 1$.

These issues and the study of the correlation functions are currently underway.

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Appendix 1

In this appendix we gather some well known results regarding normal ordering and the OPE. The reader may want to consult Mandelstam (1975), Coleman (1975), Emery (1979) and Banks *et al* (1976) for further discussion. We work in Minkowski space, but the analytical continuation to Euclidean space is straightforward.

Splitting up the fields ϕ_R and ϕ_L introduced in § 3 into creation ($\phi^-_{R,L}$) and annihilation ($\phi^+_{R,L}$) parts and using

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \tag{A1.1}$$

for $[A, B]$ commuting with A and B then the normal ordered expression is

$$:\exp(\alpha\phi_R(x)): = \exp(\alpha\phi^-_R(x)) \exp(\alpha\phi^+_R(x)) = \exp\{\tfrac{1}{2}\alpha^2[\phi^-_R(x), \phi^+_R(x)]\} \tag{A1.2}$$

with $\phi_R(x) = \phi^-_R(x) + \phi^+_R(x)$.

A similar expression is obtained for ϕ_L.

The commutator in (A1.1) is regulated by defining it as an equal time point split expression $[\phi^-_R(x), \phi^+_R(y)]|_{y=x+a}$ with a the UV cutoff (lattice spacing). The fields ϕ_R, ϕ_L commute with each other since they correspond to independent right and left going waves. Hence using equation (2.4) in the text

$$e^{\alpha\phi} = :e^{\alpha\phi}:(\mu^2 a^2)^{-\alpha^2/8\pi}$$

$$e^{\alpha\tilde{\phi}} = :e^{\alpha\tilde{\phi}}:(\mu^2 a^2)^{-\alpha^2/8\pi}.$$

By repeated use of (A1.1), it is found that

$$:\exp(i\alpha\phi_R(x))::\exp(i\beta\phi_R(y)): = :\exp[i\alpha\phi_R(x) + i\beta\phi_R(y)]:[\mu(x^+ - y^+)]^{\alpha\beta/4\pi}$$

$$:\exp(i\alpha\phi_L(x))::\exp(i\beta\phi_L(y)): = :\exp[i\alpha\phi_L(x) + i\beta\phi_L(y)]:[\mu(x^- - y^-)]^{\alpha\beta/4\pi}$$

with $x^\pm$ as defined in (3.3) and correspondingly in Euclidean space. The above equations are easily generalised for more than two normal ordered exponentials.

Appendix 2. Some OPE

Consider, for example,

$$:\cos\beta\phi(x)::\cos\beta\phi(y): = \tfrac{1}{2}:\cos\beta(\phi(x)+\phi(y)):(\mu^2|x-y|^2)^{\beta^2/4\pi} + \tfrac{1}{2}:\cos\beta(\phi(x)-\phi(y)):\left(\frac{1}{\mu^2|x-y|^2}\right)^{\beta^2/4\pi}. \tag{A2.1}$$

As $x \to y$ only the second term has singularities, writing $\phi = \phi_R + \phi_L$ and

$$\phi_R(x) - \phi_R(y) = (x^+ - y^+)\partial_y \phi_R(y)$$

$$\phi_L(x) - \phi_L(y) = (x^- - y^-)\partial_y \phi_L(y)$$

the second contribution in (A2.1), after expanding in derivatives, is

$$\tfrac{1}{2}[1 - \beta^2 |x-y|^2 \partial_y \phi_R(y) \partial_y \phi_L(y) + \text{HOD}]\left(\frac{1}{(\mu^2|x-y|^2)}\right)^{\beta^2/4\pi} \tag{A2.2}$$

By using (2.8*b*) and (3.2*b*) in Euclidean space we find

$$-2\partial_x \phi_R \partial_x \phi_L = -\tfrac{1}{2}:(\partial_x \phi)^2 + (\partial_\tau \phi)^2: + \text{constant}$$

or in Euclidean

$$-2\partial_x \phi_R \partial_x \phi_L = -\tfrac{1}{2}:(\partial_\mu \phi)^2: + \text{constant}. \tag{A2.3}$$

Now consider:

$$\cos \gamma \tilde{\phi}(x): :\cos \gamma \tilde{\phi}(y):.$$

It is written in a form similar to (A2.1). The singularity as $x \to y$ arises from the term

$$\tfrac{1}{2}:\cos \gamma(\tilde{\phi}(x) - \tilde{\phi}(y)):\left(\frac{1}{\mu^2|x-y|^2}\right)^{\gamma^2/4\pi}.$$

Now a difference arises because in $\tilde{\phi}$, ϕ_L enters with opposite sign to that in ϕ. This changes the sign in the $\partial_y \phi_R \partial_y \phi_L$ in (A2.2) (with β^2 replaced by γ^2). This change in sign is reflected in the relative sign in the beta function β_β given by equation (4.7*c*) in the text.

The OPE obtained from products of three cosine operators (of ϕ, $\tilde{\phi}$ or mixed) can be easily worked out in a similar manner.

One more OPE is necessary for the contribution of the terms: $(\partial_\mu \phi)^2$: to the renormalisation of the fugacities

$$:\partial_x \phi_R(x)::\exp(i\beta\phi_R(y)): = :\exp(i\beta\phi_R(y))\partial_x \phi_R(x): + :\exp(i\beta\phi_R(y)):\left(-\frac{i\beta}{4\pi}\right)\left(\frac{1}{x^+ - y^+}\right)$$

and similarly for ϕ_L with x^- replacing x^+.

Appendix 3. Some integrals

In the α^3 contribution to β_α (or y^3 to β_y) and G^3 to β_G we face the integrals

$$\mu^4 \int d^2y \int d^2z \left(\frac{\mu^2|x-y|^2}{(\mu^2|x-z|^2 \mu^2|y-z|^2)}\right)^{\beta^2/4\pi}$$

to this order set $\beta^2 = 8\pi$ and define $\mu(\boldsymbol{z} - \boldsymbol{x}) = \boldsymbol{R}$, $\mu(\boldsymbol{y} - \boldsymbol{z}) = \boldsymbol{S}$.

The integrals are regulated by $a < |R|$, $|S| < 1/\mu$ the integration over the relative angle is straightforward and it yields

$$\pi^2 \int_{\mu^2 a^2}^{1} d(|R|^2) \int_{\mu^2 a^2}^{1} d(|S|^2) \left(\frac{1}{|R|^4} + \frac{1}{|S|^4} + \frac{4}{|R|^2|Y|^2}\right).$$

The first two terms are disconnected since they correspond to $y \to z$ ($|x| > |z|$) and $x \to z$ ($|y| > |z|$), respectively, they must be discarded. Only the third term is connected.

In the YG^2 contribution to β_Y and the GY^2 contribution to β_G we face the integrals

$$\mu^4 \int d^2y \int d^2x \left(\frac{1}{\mu^2|x-y|^2}\right)^{\beta^2/4\pi} \left(\frac{(z^+ - x^+)(z^- - y^-)}{(z^- - x^-)(z^+ - y^+)}\right)^{\beta^2/4\pi}$$

writing $x - y = \boldsymbol{R}$, $y - z = \boldsymbol{S}$, $R^+S^- = |R||S|e^{i\Theta}$ with Θ the relative angle between $\boldsymbol{R}$ and $\boldsymbol{S}$.

To this order set $\beta^2 = 8\pi$. The angular integration is easily performed by expanding the bracket in angular momentum states ($e^{il\theta}$). To do this, the integral over $|\boldsymbol{S}|$ must be split into two domains: $|\boldsymbol{R}| > |\boldsymbol{S}|$ and $|\boldsymbol{R}| < |\boldsymbol{S}|$.

In the first there are no terms surviving the angular integration. The second yields

$$\pi^2 \int_{\mu^2a^2}^{1} \frac{d(|R|^2)}{|R|^4} \int_{|R|^2}^{1} d(|S|^2)\left(1 - 4\frac{|R|^2}{|S|^2} + 3\frac{|R|^4}{|S|^4}\right).$$

Again there is a disconnected piece of the form

$$\int_{\mu^2a^2}^{1} \frac{d(|R|^2)}{|R|^4}$$

that must be subtracted.

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